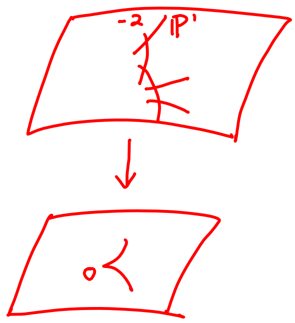
Nakajima (ICM) Geometric Construction of Representations of Affine Algebras.

§1. finite $\Gamma \lneq SL(2,\mathbb{C})$.

$\left(\Leftrightarrow \Gamma \lneq \underset{\text{CY2}}{SU(2)} = \underset{\text{HK}}{Sp(1)} \overset{/\mathbb{Z}_2}{\Longleftrightarrow} \text{finite group of isometry of } \mathbb{R}^3 \right)$

minimal resol$^{\underline{n}}$. $\begin{array}{c} M \\ \downarrow \pi \\ \mathbb{C}^2/\Gamma \end{array}$



- irred. comp. of $\pi^{-1}(0) \longleftrightarrow$ vertices of ADE Dynkin diagram.
- $H_2(\pi^{-1}(0), \mathbb{C}) \cong \mathfrak{h}$
- Grothendieck / Brieskorn / Slodowy

  $\mathbb{C}^2/\Gamma$, deform, resol$^{\underline{n}}$ $\leftsquigarrow \mathfrak{g}$
- McKay $\Gamma \lneq SL(2,\mathbb{C}) \curvearrowright \mathbb{C}^2 = Q$

  $Q \otimes \rho_i = \bigoplus_j a_{ij}\, \rho_j \rightsquigarrow$ $\hat{A}\hat{D}\hat{E}$ Dynkin diagram

  irred. Γ-mod: ρ_i's $\longleftrightarrow$ vertices

  $a_{ij} \longleftrightarrow$ edge
- Gonzalez-Sprinberg, Verdier

$$R(\Gamma) = K_\Gamma(\mathbb{C}^2) \xleftarrow{\sim} K(M)$$

Role of $\hat{\mathfrak{g}}$?

§2. • Kronheimer

quiver $\rightsquigarrow$ $\mathbb{C}^2/\Gamma$, deform, resol^{n}.

• (linear alg. $\overset{\text{ADHM}}{\rightsquigarrow}$ $\{\text{ASD}/\mathbb{C}^2\}$)

• Kronheimer - Nakajima

quiver $\overset{\text{ADHM}}{\rightsquigarrow}$ $\mathcal{M} = \{\text{ASD bdl}/M\}$

• Different viewpoint :

$$\begin{array}{ccc} \mathrm{Hilb}^n(\mathbb{C}^2) & & \overset{\text{codim } n}{I \triangleleft \mathbb{C}[x,y]} \\ \pi \downarrow & \text{Hilbert-Chow morphism.} & \\ S^n(\mathbb{C}^2) = (\mathbb{C}^2)^n/S_n & & \end{array}$$

(Fogarty) resol^{n} of sing.

(Beauville) holo. sympl.

(Nakajima) hyper Kähler

$$\begin{array}{ccc} \Gamma \curvearrowright \mathrm{Hilb}^n(\mathbb{C}^2) & \supseteq & \mathrm{Hilb}^n(\mathbb{C}^2)^\Gamma \\ \downarrow & & \\ \Gamma \curvearrowright S^n(\mathbb{C}^2) & \supseteq & (S^n(\mathbb{C}^2))^\Gamma = S^m(\mathbb{C}^2/\Gamma) \end{array}$$

$m = \left[\frac{n}{|\Gamma|}\right]$, multi(origin) $= n - m|\Gamma|$.

$I \in \mathrm{Hilb}^n(\mathbb{C}^2)^\Gamma \Rightarrow \Gamma \curvearrowright \mathbb{C}[x,y]/I$

$\forall\ \Gamma$-mod V, $\underset{\substack{4 \\ \text{Hyperkähler (Crawley-Boevey)}}}{M(V)} := \left\{ I \in \mathrm{Hilb}^n(\mathbb{C}^2)^\Gamma : \frac{\mathbb{C}[x,y]}{I} \simeq V \right\}$

Eg. $V = \mathbb{C}[\Gamma]$ regular rep.. $\dim V = |\Gamma| =: n$

$\rightsquigarrow (S^{|\Gamma|}\mathbb{C}^2)^\Gamma \simeq \mathbb{C}^2/\Gamma \subseteq S^n\mathbb{C}^2$

space of Γ-orbits

- $(\mathrm{Hilb}^{|\Gamma|}(\mathbb{C}^2))^\Gamma \simeq M$, π = resol^n.

§3. $\mathbb{C}^2 \subset \mathbb{P}^2 = \mathbb{C}^2 \amalg l_\infty$

$\mathcal{M}_{/\mathbb{P}^2}(\underbrace{n}_{c_2}, \underbrace{r}_{r}) = \text{moduli}\left\{ \begin{array}{l} \text{torsion-free shf}/\mathbb{P}^2 \\ E|_{l_\infty} \xrightarrow[\varphi]{\sim} \mathcal{O}_{\mathbb{P}^1}^{\oplus r} \end{array} \right\}$ ← framing ⇓ $c_1(E) = 0$

- q-proj., HK.

$0 \to E \to \underset{VB}{E^{\vee\vee}} \to \mathcal{O}_Z \to 0$

$$\begin{array}{lcl} \text{Gieseker} & \mathcal{M}(n,r) & E \\ & \downarrow & \downarrow \\ & & (E^{\vee\vee}, \operatorname{supp} E^{\vee\vee}/E) \\ \text{Uhlenbeck} & \mathcal{M}_0(n,r) = \bigsqcup_{n'+n''=n} \underbrace{\mathcal{M}_0^{reg}(n',r)}_{\{\text{loc. free}\}} \times S^{n''}\mathbb{C}^2 & \leftarrow \text{affine alg. var.} \end{array}$$

Eg. $$\begin{array}{ccc} \mathcal{M}(n,1) & = & \mathrm{Hilb}^n\mathbb{C}^2 \\ \downarrow & & \\ \mathcal{M}_0(n,1) & = & S^n\mathbb{C}^2 \end{array}$$

- $SL(2,\mathbb{C}) \geqslant \Gamma \overset{\text{fix}}{\curvearrowright} \mathcal{O}_{l_\infty}^{\oplus r} = W \underset{\mathbb{C}}{\otimes} \mathcal{O}_{l_\infty}$, $W \in \Gamma$-mod.

$\rightsquigarrow \Gamma \curvearrowright \mathcal{M}(n,r) \supseteq \mathcal{M}(n,r)^\Gamma \overset{\text{write}}{=\!=} \bigsqcup_V \underbrace{\mathcal{M}(V,W)}_{HK.}$

$$\begin{array}{ccc} \mathcal{M}(n,r) & & \\ \downarrow & & \\ \mathcal{M}_0(n,r) & \supseteq & \mathcal{M}(n,r)^\Gamma \end{array}$$

- $\mathcal{M}(V,W) \triangleq \{(E,\varphi) : V = H^1(\mathbb{P}^2, E(-1))$ as Γ-mod$\}$

(e.g. $E = \mathcal{I}_Z \Rightarrow V = H^1(\mathbb{P}^2, \mathcal{I}_Z(-1)) \simeq H^0(\mathbb{P}^2, \mathcal{O}_Z(-1)) \simeq \frac{\mathbb{C}[x,y]}{I_Z}$)

- These are all quiver var. of affine types.
  as cpx. parameter $\to 0$

  cpx. parameter $\neq 0 \rightsquigarrow \{$instantons / non-comm. $\mathbb{P}^2\}$.

§4. $$\mathcal{M}_0(n,r)^\Gamma = \bigsqcup_{n''+|v^0|=n} \mathcal{M}_0^{reg}(v^0, w) \times (S^{n''}\mathbb{C}^2)^\Gamma$$

where $$(S^{n''}\mathbb{C}^2)^\Gamma = \bigsqcup_{m\leq n''} \underbrace{S^m_\lambda(\mathbb{C}^2/\Gamma)}_{\sum_{i=1}^{r}\lambda_i[x_i] \in S^m(\mathbb{C}^2/\Gamma)}$$

$\lambda = (\lambda_1, \dots, \lambda_r)$ partition of m

$x_i \neq x_j \neq 0$.

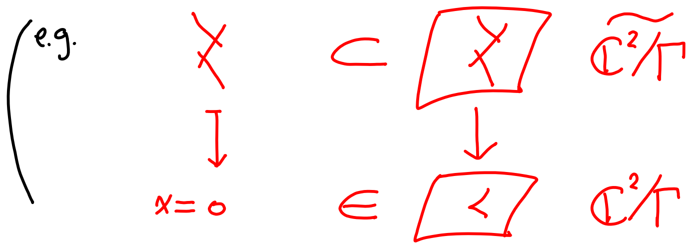
$n'' - m =$ multi of cycle at $0 \in \mathbb{C}^2/\Gamma$

$$\begin{array}{ccc} \mathcal{M}(n,r)^\Gamma = \bigsqcup_v \mathcal{M}(v,w) & \supset & \bigsqcup_v \mathcal{M}(v,w)_x = \pi^{-1}(x). \\ \downarrow \pi & & \downarrow \\ \mathcal{M}_0(n,r)^\Gamma & \ni & x \end{array}$$

- $x = (W \underset{\mathbb{C}}{\otimes} \mathcal{O}_{\mathbb{P}^2}, \varphi, 0)$, write

  $\mathcal{M}(v,w)_x =: \mathcal{L}(v,w) \underset{\text{Lagr.}}{\subseteq} \mathcal{M}(v,w)$ central fiber

  (e.g.



- $\forall\ x \quad \exists\ v_s, w_s$ s.t.

  $$\mathcal{M}(v,w)_x = \mathcal{L}(v_s, w_s) \times \prod^{x_i} \mathrm{Hilb}_0^{\lambda_i}(\mathbb{C}^2)$$
  pure dim.

§5 Fix $\Gamma \curvearrowright W$ $\quad$ w its isom. class.

Sum over v's

$$\mathfrak{M}(w) \triangleq \coprod_n \mathfrak{M}(n,r)^\Gamma = \coprod_v \mathfrak{M}(v,w) \supset \coprod_v \mathfrak{L}(v,w) =: \mathfrak{L}(w)$$

$$\Big\downarrow$$

$$\mathfrak{M}_0(\infty, w) \triangleq \bigcup_n \mathfrak{M}_0(n,r)^\Gamma$$

$$Z(w) \triangleq \mathfrak{M}(w) \underset{\mathfrak{M}_0(\infty,w)}{\times} \mathfrak{M}(w) \quad \underset{\text{cx. Lagr.}}{\subset} \mathfrak{M}(w) \times \mathfrak{M}(w)$$

Convolution product

$$H_{top}(Z(w)) \otimes H_{top}(Z(w)) \xrightarrow{*} H_{top}(Z(w)) \qquad \text{coeff: } \mathbb{C}$$

$$p_{13*}(p_{12}^*(-) \cap p_{23}^*(-))$$

$$\rightsquigarrow \underset{\text{assoc. alg. w/ 1.}}{(H_{top}(Z(w)), *)} \curvearrowright H_{top}(\underbrace{\mathfrak{M}(w)_x}_{\pi^{-1}(x)}) \quad \forall x \in \mathfrak{M}_0(\infty, w)$$

(Nakajima) $\quad \exists \; U(\hat{\mathfrak{g}}) \to H_{top}(Z(w)) \curvearrowright H_{top}(\mathfrak{M}(w)_x)$

irred. highest weight integrable $U(\hat{\mathfrak{g}})$-mod.

w/ wt. spaces $= H_{top}(\mathfrak{M}(v,w)_x)$

Construction:

$h_i, d \mapsto$ fund. classes of diagonals.

$$e_i \mapsto \coprod_v \left\{ \begin{matrix} (E,\varphi) \\ \times \\ (E',\varphi') \end{matrix} \in \begin{matrix} \mathfrak{M}(v,w) \\ \times \\ \mathfrak{M}(v+\rho_i, w) \end{matrix} \;\middle|\; E \subset E' \right\}$$

$f_i \mapsto$ swap 2 factors.

Eg. $\mathcal{L}(w)$ case : ht wt vector $= [\mathcal{L}(0,w)]$
$\quad\in$ 1 pt $\sim E = W \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^2}$
wt. $H_{Top}(\mathcal{L}(v,w)) = w - v$.

Special case : $w_0 = \Lambda_0$ 0th. fund. wt.
$\rightsquigarrow \bigoplus_{v_0=1} H_{Top}(M(v), \mathbb{C}) \simeq \mathfrak{g}$ ← finite dim ADE

Cor. irred. comp. $\mathfrak{M}(w)_x$'s $\rightsquigarrow$ basis of $H_{Top}(\mathfrak{M}(w)_x)$
crystal (Kashiwara), semi-canonical.

How about $\hat{\mathfrak{g}} \curvearrowright H_{Top - \underset{\sim}{d}}(\mathfrak{M}(w)_x)$?
$\Vert$ Decomposition Thm.
$$\bigoplus_y H^{d + \dim \mathcal{O}_x}(i_x^! IC(\mathcal{O}_y)) \otimes \underbrace{H_{Top}(\mathfrak{M}(w)_y)}_{\mathfrak{g} \text{ acts.}}$$
$i_x : \{x\} \hookrightarrow \mathfrak{M}_0(\infty, w)$

($\because$ $\mathfrak{M}(w) \xrightarrow{\pi} \mathfrak{M}_0(\infty, w)$ semi-small.
i.e. $\pi|_{\pi^{-1}(\mathcal{O}_x)}$: topo. fiber bdl. & $2 \dim \mathfrak{M}(w)_x \leq \operatorname{codim} \mathcal{O}_x$
$\quad\in$ stratum)

Remark: When $w = \rho_0$
($\Rightarrow$ every stratum $\mathcal{O}_y = S^k(\mathbb{C}^2/\Gamma)$ only quot. sing.
$IC(\mathcal{O}_y) \simeq \underline{\mathbb{C}}$)
$H_*(\mathcal{L}(w))$ "Fock space"

§6 $H_{Top} \rightsquigarrow K$-groups. (2dTFT $\rightsquigarrow$ 3d TFT)

$L\hat{\mathfrak{g}} = \hat{\mathfrak{g}} \otimes_{\mathbb{C}} \mathbb{C}[z^{\pm}]$ ~ double loop of $\mathfrak{g}$

(via Drinfeld realizat^{n} of quantum loop alg)

$\rightsquigarrow$ quantum toroidal alg. $U_q(L\hat{\mathfrak{g}})$

- $\underbrace{\mathrm{Aut}_{\Gamma}(W)}_{G_W} = \prod_i GL(w_i, \mathbb{C})$

$$\begin{tikzcd} G_W \arrow[r, phantom, "\curvearrowright"] \arrow[dr, "\text{change frame}"'] & \mathcal{M}(w) \arrow[d, "\pi"] & \mathbb{C}^\times \arrow[l, phantom, "\curvearrowleft"] \arrow[dl] \\ & \mathcal{M}_0(\infty, w) & \end{tikzcd}$$

$\mathbb{C}^\times$ scaling

$\mathbb{C}^\times \curvearrowright (\mathbb{P}^2, \ell_\infty)$

(Nakajima) $\mathcal{U}_q^{\mathbb{Z}}(L\hat{\mathfrak{g}}) \xrightarrow{\exists} K^{G_W \times \mathbb{C}^\times}(Z(w))$ mod. torsion

$\wedge|$

$\mathcal{U}_q(L\mathfrak{g})$

$\mathbb{Z}[q^{\pm}]$-alg. homo.

(Constr: Similar as before.
$e_i \otimes z^r \mapsto$ line bdl. w/ fiber/ $\frac{(E,\varphi)}{(E',\varphi')}$ as $H^0(E'/E)^{\otimes r}$

Application: $\rightsquigarrow$ ℓ-integ. rep. of $\underbrace{U_q^{\mathbb{Z}}(L\hat{\mathfrak{g}})|_{q=\varepsilon}}_{U_\varepsilon(L\hat{\mathfrak{g}})} \in \mathbb{C}^\times$

i.e. $K^{G_W \times \mathbb{C}^\times}(\mathcal{L}(W)) \otimes_{R(G_W \times \mathbb{C}^\times) \ni ev.} \mathbb{C}$, character formula ✓

Drinfeld-Chari-Pressley classify (finite dim) $U_\varepsilon(L\mathfrak{g})$-mod

(Nakajima) ℓ-integ. $U_\varepsilon(L\hat{\mathfrak{g}})$-mod;

$$H_*(\mathcal{M}(w)^A_x) = \sum_y H^*(i_x^! IC(\mathcal{O}_y)) \otimes L_y$$

dim=?

unique irred. quot. of $H_*(\mathcal{M}(w)^A_y)$.

(reason: $K \xrightarrow{ch} H^\cdot$ and localization)

Nakajima, Homology of moduli spaces of instatons on ALE spaces, I

section 2.

§ Review constr. ALE & ADHM.

$V, W \curvearrowleft \Gamma \leqslant SU(2) \curvearrowright \mathbb{C}^2 = Q$

$$/\mathbb{C}_I \qquad M := \mathrm{Hom}_\Gamma(V, Q\otimes V) + \mathrm{Hom}_\Gamma(W,V) + \mathrm{Hom}_\Gamma(V,W)$$

$$\ni (B_1, B_2, i, j)$$

anti-linear $J \downarrow$

$$(-B_2^+, B_1^+, -j^+, i^+)$$

$U(V)^\Gamma \curvearrowright (M; I, J, K=IJ)$ Hypenkähler

$\rightsquigarrow$ HK moment $\xrightarrow{\mu = (\mu_\mathbb{R}, \mu_\mathbb{C})} u(V)^\Gamma + gl(V)^\Gamma$

$$\mu_\mathbb{R} = \tfrac{i}{2}\left([B_1, B_1^+] + [B_2, B_2^+] + i i^+ - j^+ j\right)$$

$$\mu_\mathbb{C} = [B_1, B_2] + ij$$

Decompose into Γ-irred. $\underset{\text{trivial}}{R_0}, \dots, R_n$

$$V = \Sigma V_k \otimes R_k \qquad W = \Sigma W_k \otimes R_k$$

$$\Rightarrow \mathrm{Hom}_\Gamma(V, Q\otimes V) = \bigoplus \mathrm{Hom}(V_\ell, V_k) \otimes \underbrace{\mathrm{Hom}_\Gamma(R_\ell, Q\otimes R_k)}_{\dim = a_{k\ell} = 0,1 \rightsquigarrow \text{Dynkin}}$$

$$\Rightarrow M = \bigoplus_{\substack{\ell \to k \\ (\text{i.e. } a_{k\ell}=1)}} \mathrm{Hom}(V_\ell, V_k) + \bigoplus_m \left(\mathrm{Hom}(W_m, V_m) + \mathrm{Hom}(V_m, W_m)\right)$$

$$(B_{k,\ell}, i_m, j_m)$$

Fix an orientat$^{\text{n}}$ of Dynkin diagram, $\varepsilon(k,\ell) = \pm 1$.

$$\mu_\mathbb{R} = \tfrac{i}{2}\left(\sum_{k\to\ell} B_{k,\ell} B_{k,\ell}^+ - B_{\ell k}^+ B_{\ell,k} + i_k i_k^+ - j_k^+ j_k\right)$$

$$\mu_\mathbb{C} = \sum_{k\to\ell} \varepsilon(k,\ell) B_{k,\ell} B_{\ell,k} + i_k j_k \in \bigoplus_k gl(V_k)$$

Choose $\zeta = (\zeta_{\mathbb{R}}, \zeta_{\mathbb{C}}) \in Z_V \oplus (Z_V \otimes \mathbb{C})$ ← center of $U(V)^\Gamma$

$$\mathfrak{M} \triangleq \{ (B, i, j) \in M \mid \underbrace{\mu = -\zeta}_{\text{ADHM-eqt.}} \} / U(V)^\Gamma$$

$\mathfrak{M}^{\text{reg.}}$ w/ trivial stabilizer. ← non-sing. HK mfd.

$$\dim_{\mathbb{R}} \mathfrak{M} = 2 \vec{v}^t (2\vec{w} - \tilde{C}\vec{v})$$

$\vec{v} = (\dim V_0, \cdots, \dim V_n)$
$\vec{w} = (\dim W_0, \cdots, \dim W_n)$
$\tilde{C}$: extended Cartan matrix.

Eg. $V = \overbrace{L^2(\Gamma)}^{R}$, $W = 0$ (eg. 1 2 $\overset{1}{\underset{3}{2}}$ 2 1)

$= \sum \mathbb{C}^{n_k} \otimes R_k$ $\quad n_k = \dim R_k$

$$X := \{ y \in M : \mu(y) = \zeta \} / PU(R)^\Gamma$$

$\quad$ ← $U(1)$-act trivially.

(Kronheimer) ζ generic

$\Rightarrow$ $PU(R)^\Gamma \curvearrowright \mu^{-1}(\zeta)$ free

X ALE HK^4 mfd. on min. resol^{n} of $\mathbb{C}^2/\Gamma$

· naturally principal bdl. $\underbrace{PU(R)^\Gamma}_{\prod_{k \neq 0} U(n_k)} \longrightarrow \mu^{-1}(\zeta)$

$$\mu^{-1}(\zeta) \downarrow X$$

⇝ assoc. VB $\mathbb{C}^{n_k} \longrightarrow \mathcal{R}_k \longrightarrow X$ $\quad k = 1, \cdots, n$

· Given $V, W \curvearrowleft \Gamma$

$$(B, i, j) \in \mu^{-1}(-\zeta) \subset M(\vec{v}, \vec{w})$$

$$\rightsquigarrow \sum_\ell V_\ell \otimes \mathcal{R}_\ell \xrightarrow{\sigma} \begin{matrix} \bigoplus_{\ell \to k} V_k \otimes \mathcal{R}_\ell \\ + \\ \bigoplus_\ell W_\ell \otimes \mathcal{R}_\ell \end{matrix} \xrightarrow{\tau} \bigoplus_\ell V_\ell \otimes \mathcal{R}_\ell$$

$$\begin{pmatrix} B_{k,\ell} \otimes 1 \pm 1 \otimes \xi_{k,\ell} \\ j_\ell \otimes 1 \end{pmatrix} \qquad (\pm B_{\ell,k} \otimes 1 - 1 \otimes \xi_{\ell,k} \quad i_\ell \otimes 1)$$

$\xi_{k,\ell}$ ← tauto. VB homo.

$\mu_{\mathbb{C}}(B, i, j) = -\zeta_{\mathbb{C}} \Rightarrow \tau\sigma = 0$ i.e. complex.

trivial stabilizer $\Rightarrow$ σ 1-1 & τ onto.

cohomology: $E \triangleq \mathrm{Coker}(\sigma, \tau^{+}) \subseteq \begin{array}{c} \bigoplus_{l \to k} V_k \otimes R_l \\ + \\ \bigoplus_{l} W_l \otimes R_l \end{array}$

(Kronheimer-Nakajima) · E ASD.

$A - A_0 = O(r^{-3})$ etc.

- all such ASD conn. is obtained this way. ($\vec{v}, \vec{w}$ are determined by E_{top} & A_0).

- $\bar{E} \longrightarrow \bar{X} = X \cup \infty$ $\rightsquigarrow$ $\Gamma \curvearrowright \bar{E}_\infty$

  orbi VB   orbifold.

- $\mathcal{M}$ = frame moduli space of ASD on $\bar{E}$

§ A_n-case. $S^1 \curvearrowright \mathcal{M}$ $\rightsquigarrow$ $H_*(\mathcal{M})$ ✓

Note: W-part is just add on. Say $W=0$.

$$\rightsquigarrow \sum_l V_l \otimes R_l \xrightarrow{\sigma} \begin{array}{c} \bigoplus_{l \to k} V_k \otimes R_l \\ + \\ \bigoplus_{l} W_l \otimes R_l \end{array} \xrightarrow{\tau} \bigoplus_l V_l \otimes R_l$$

$$\chi = \sum_l n_l \Big(\sum_{k \sim l} v_k - 2 v_l \Big)$$

Nakajima, Instantons/ALE, Quiver var., Kac-Moody.

§2. Quiver variety

Given graph $\bullet \overset{V_i}{\underset{W_i}{\text{---}\bullet\text{---}}} \bullet$ ← Herm. vector space

$H := \{ \underset{out(h)}{\bullet} \xrightarrow{h} \underset{in(h)}{\bullet} \subset \text{graph} \} = \Omega \sqcup \overline{\Omega}$ & $\Omega \cap \overline{\Omega} = \phi$

↑reverse orientation.

Assume $\nexists$ cycle $\in \Omega$

$$\underset{\prod_k U(V_k)}{\overset{\|}{G_V}} \curvearrowright \mathbb{M} := \bigoplus_{h \in H} \mathrm{Hom}(V_{out(h)}, V_{in(h)}) + \bigoplus_{k=1}^{n} \begin{matrix} \mathrm{Hom}(W_k, V_k) \\ + \\ \mathrm{Hom}(V_k, W_k) \end{matrix}$$

$$\text{H.K.} \quad = \mathbb{M}_\Omega \oplus \mathbb{M}_{\overline{\Omega}} = T^* \underbrace{\mathbb{M}_{\overline{\Omega}}}_{\text{cx. Lagr.}}$$

Quiver variety $\mathcal{M}_\zeta := \mathbb{M} /\!/\!/ G_V$ ← HK.

cpx ADHM eqt. $\Rightarrow$ $\tau\sigma = 0$ $\rightsquigarrow$ holo bdl.
real ADHM eqt. $\Rightarrow$ ASD bdl.

§4. $\exists$ $\mathcal{M}_{(\zeta_{\mathbb{R}}, \zeta_{\mathbb{C}})} \longrightarrow \mathcal{M}_{(0, \zeta_{\mathbb{C}})}$

- I-holo.
- resol^{n} of singularities

§5 $S^1 \curvearrowright \bigsqcup_{\zeta_{\mathbb{C}} \in Z\otimes\mathbb{C}} \mathcal{M}_{(\zeta_{\mathbb{R}},\zeta_{\mathbb{C}})}$

$$\downarrow$$

$S^1 \curvearrowright Z\otimes\mathbb{C}$

$t\cdot(B_h, i, j) = (t^{\frac{1-\varepsilon(h)}{2}} B_h, i, t j)$

$\mu_{\mathbb{R}}(t\cdot(-)) = \mu_{\mathbb{R}}(-)$

$\mu_{\mathbb{C}}(t\cdot(-)) = t\,\mu_{\mathbb{C}}(-)$

preserve I & metric

$\omega_{\mathbb{C}} \mapsto t\,\omega_{\mathbb{C}}$

- moment map $F = \sum_{h\in\Omega} |B_h|^2 + \sum |j|^2$ is proper.
- extend to $\mathbb{C}_I^\times \curvearrowright \bigsqcup_{\zeta_{\mathbb{C}}} \mathcal{M}_{(\zeta_{\mathbb{R}},\zeta_{\mathbb{C}})} \xrightarrow{\pi} \bigsqcup_{\zeta_{\mathbb{C}}} \mathcal{M}_{(0,\zeta_{\mathbb{C}})}$

coincide w/ Slodowy's action.

- $\mathbb{C}_I^\times \curvearrowright \mathcal{M}_{(\zeta_{\mathbb{R}},0)} \xrightarrow{\pi} \mathcal{M}_{(0,0)}$

$$\underbrace{\mathbb{C}_I^\times\text{-fix pt.}}_{\substack{\mathfrak{F} = \bigsqcup_{\alpha=1}^{p} \mathfrak{F}_\alpha \\ \text{smooth}}} \subset \underbrace{\pi^{-1}(0)}_{\{\text{nilp. elt.}\}} =: \mathcal{L} \underset{\substack{\varphi \\ \text{h.e.}}}{\subset} \mathcal{M}_{(\zeta_{\mathbb{R}},0)}$$

$$\mathcal{L} = \bigsqcup_{\alpha=1}^{p} \left\{ x \in \mathcal{M}_{(\zeta_{\mathbb{R}},0)} \;\middle|\; \exists \lim_{t\to 0} t\cdot x \in \mathfrak{F}_\alpha \right\}$$

$$= \left\{ (B,i,j) \in \mathcal{M}_{(\zeta_{\mathbb{R}},0)} \;\middle|\; i = 0,\ \overline{G_{\vec{v}}^{\mathbb{C}}\cdot B} \ni 0 \right\} \quad \text{if } \zeta_{\mathbb{R}} \text{ 'good'}$$

§7. Partial flag variety (A_{r-1})

$\mathfrak{F} = \mathfrak{F}(v_1 > \cdots > v_n ; r)$

$$T^*\mathfrak{F} = \left\{ (\phi, A) \in \mathfrak{F}\times\mathfrak{gl}(r) : \begin{array}{l} \phi = (E^0 \supset E^1 \supset \cdots \supset E^n) \\ A(E^k) \subset E^{k-1} \quad \forall k \end{array} \right\}$$

$\overset{1}{\bullet} \to \overset{2}{\bullet} \to \bullet \cdots \overset{n}{\bullet}$ $\qquad \vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \qquad \vec{w} = \begin{pmatrix} r \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

Thm. $\mathcal{M}_{(\zeta_{\mathbb{R}},0)} \cong T^*\mathfrak{F}$

(Kraft-Procesi): $\mathcal{M}_{(0,0)} \longrightarrow \mathfrak{gl}(r)$: $(B,i,j) \mapsto j_1 \circ i_1$

emb. onto $\overline{GL(r)\cdot X}$ $\exists$ nilp. $X \in \mathfrak{gl}(r)$

Pf. of thm: Using $\mathbb{R}$-ADHM eqt.

$$\mathcal{M}_\xi \simeq \{(B, i, j) \mid \mu_{\mathbb{C}} = 0,\ \underset{\forall k \geq 2}{B_{k-1,k}},\ j: 1\text{-}1\} / G_V^{\mathbb{C}}$$

$$\xrightarrow{\sim} T^*\mathcal{F} \subseteq \mathcal{F} \times \mathfrak{gl}(r)$$

$$(\phi = (E^0 \supset E^1 \supset \cdots \supset E_n),\ A)$$

w/ $E^k = \mathrm{Im}(j_1 B_{12} B_{23} \cdots B_{k-1,k})$ $\quad A = j_1 \circ i_1$

($\mathbb{C}$-ADHM eqt. $\Rightarrow$ lie inside $T^*\mathcal{F}$)

§ 8. A_{r-1}-case

- $\mathcal{M}_0^{reg}(\vec{v}, \vec{w}) \neq \phi \Rightarrow \vec{u} := \vec{w} - C\vec{v} \in \mathbb{Z}^n_{\geq 0}$

Consider partition of r

$$r = \sum_{k=1}^{n} k\, w_k \qquad \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n),\quad \sum \lambda_i = r$$

$$\lambda = (\underbrace{n \cdots n}_{w_n}, \cdots, \underbrace{1\ 1 \cdots 1}_{w_1})$$

Define another partition of r

$$\mu = (\mu_1 \geq \cdots \geq \mu_{n+1})$$

by $\mu_i = v_n + \sum_{k \geq i} u_k$ & $\mu_{n+1} = v_n$

	nilp. matrix$_{r\times r}$	size of Jordan cells	conj. class.	transversal slice
N_λ	$\in \mathcal{N}(\mathfrak{gl}(r))$	$\lambda_1, \ldots, \lambda_n$	$\mathcal{O}_\lambda$	S_λ
N_μ	— " —	$\mu_1, \ldots, \mu_{n+1}$	$\mathcal{O}_\mu$	S_μ

(Kronheimer) $\quad \mathcal{M}^{reg}_{(0,0)}(\vec{v}, \vec{w}) = \mathcal{O}_\mu \cap S_\lambda$

- Viewed as $\{SU(2)\text{-equivan. ASD conn.}/\mathbb{R}^4\}/\cong$
- $\mathcal{M}_0^{reg} \neq \phi \iff \mathcal{O}_\lambda \subseteq \overline{\mathcal{O}_\mu}$
- $\mathcal{M}_0 = \overline{\mathcal{O}_\mu} \cap S_\lambda$
- If $\vec{w} = \begin{pmatrix} r \\ 0 \\ \vdots \\ 0 \end{pmatrix} \Rightarrow S_\lambda = \mathfrak{gl}(r) \Rightarrow \mathcal{M}_0 = \overline{\mathcal{O}_\mu}$ $\quad$ nilpotent orbits.

$$\mathfrak{M}_{(\zeta_{\mathbb{R}},0)}(\vec{v},\overset{\vec{w}}{\overset{\shortparallel}{r}}) = T^*\mathcal{F}(\vec{v},r)$$
$$\downarrow \pi \qquad \text{proj.}$$
$$\mathfrak{gl}(r)$$

$\mathrm{Image}(\pi) = \overline{\mathcal{O}_\mu}$

$\pi \rightsquigarrow$ resol$^{\text{n}}$ of sing. of $\overline{\mathcal{O}_\mu}$

Conj. : True $\forall\, \vec{w}$.

§9 ADE-graph $\quad \sigma \in W$

$$\Rightarrow \quad \mathfrak{M}_\zeta(\vec{v},\vec{w}) \xrightarrow[\simeq]{\Phi_\sigma} \mathfrak{M}_{\sigma\zeta}(\vec{v}',\vec{w})$$

$$\sigma(\vec{w}-C\vec{v}) = \vec{w}-C\vec{v}'$$

$$\Phi_\sigma \circ \Phi_\tau = \Phi_{\sigma\tau}$$

Cor. $\vec{w}-C\vec{v}=0 \Rightarrow W \curvearrowright H^\bullet(\mathfrak{M}_\zeta(\vec{v},\vec{w});\mathbb{Z})$

$(\Rightarrow \vec{v}'=\vec{v} \ \ \forall \sigma)$

§10. Repr. of Kac-Moody algebras.

C gen. Cartan $\rightsquigarrow$ Kac-Moody $\mathfrak{g} = \langle H_k, E_k, F_k \rangle_{k=1}^n$

univ. enveloping alg. $\mathcal{U} \supset \mathcal{U}^- = \langle F_k\text{'s} \rangle$

$\mathfrak{g} \curvearrowright L(w)$ irred ht. wt w integ. repr.

$$0 \longrightarrow \langle F_k^{w_k+1}\text{'s} \rangle \longrightarrow \mathcal{U}^- \longrightarrow L(w) \longrightarrow 0$$
$$\alpha \longmapsto \alpha \cdot x$$

x ← ht. wt. vector

$$M(X) := \{ f : X \xrightarrow{\text{constructible}} \mathbb{Q} \}$$

$$X \xrightarrow{p} X' \qquad M(X) \underset{p_!}{\overset{p^*}{\rightleftarrows}} M(X')$$

$$(p_! f)(x) = \sum_{a\in\mathbb{Q}} a\, \chi(p^{-1}(x) \cap f^{-1}(a))$$

$$\Lambda(v,w) = \{ (B,0,j) \in \mu_{\mathbb{C}}^{-1}(0) \mid 0 \in \overline{G_V^{\mathbb{C}} \cdot B} \}$$

$$v = v' + v'' \quad \& \quad w \quad \in \mathbb{Z}^n_{\geqslant 0}$$

$$\begin{tikzcd}
\left\{\begin{array}{l} \text{same as right } \& \\ V'_k \xrightarrow[\simeq]{R'_k} C_k \\ V''_k \xrightarrow[\simeq]{R''_k} V_k/C_k \end{array}\right\} \arrow[r, "\text{forget}"', "{R',\, R''}" below, "p_2"] \arrow[d, "p_1"] & \left\{\begin{array}{l} (B, 0, j) \in \Lambda(v, w) \\ (C_k \leq V_k\text{'s}) \ B\text{-stable} \\ (\dim(C_k\text{'s}) = v - v'' \\ \qquad = v') \end{array}\right\} \arrow[d, "p_3"', "\text{forget } C_k\text{'s}"] \\
\Lambda(v'', 0) \times \Lambda(v', w) & \Lambda(v, w)
\end{tikzcd}$$

$(B'', 0, 0)$, $(B', 0, j')$

$$\begin{tikzcd}
V'_{out(h)} \arrow[r, "R'_{out(h)}", "\sim"'] \arrow[d, dashed, "B'_h"'] & C_{out(h)} \arrow[d, "B_h"] \\
V'_{in(h)} \arrow[r, "R'_{in(h)}", "\sim"'] & C_{in(h)}
\end{tikzcd}
\qquad
\begin{tikzcd}
V'_k \arrow[r, "R'_k", "\sim"'] \arrow[dr, dashed, "j'_k"'] & C_k \arrow[d, "j_k"] \\
 & W_k
\end{tikzcd}
\qquad
\begin{tikzcd}
V''_{out(h)} \arrow[r, "R''_{out(h)}", "\sim"'] \arrow[d, dashed, "B''_h"'] & V_{out(h)}/C_{out(h)} \arrow[d, "B_h"] \\
V''_{in(h)} \arrow[r, "R''_{in(h)}", "\sim"'] & V_{in(h)}/C_{in(h)}
\end{tikzcd}$$

Check: B', B'' nilp.

Restrict to $H^s_\zeta \ni (B, i, j)$ & $/G^{\mathbb{C}}_v$

Ω (i) (moment) $\mu_{\mathbb{C}}(B, i, j) = -\zeta_{\mathbb{C}}$, $G^{\mathbb{C}}_v \cdot (B, i, j) \cap \mu_{\mathbb{R}}^{-1}(-\zeta_{\mathbb{R}}) \neq \emptyset$

(ii) (stability) $\forall\, S_k \leq V_k$'s w/ $B_h(S_{out(h)}) \subset S_{in(h)}$ $j_k(S_k) = 0 \Rightarrow S_k\text{'s} = 0$

(iii) $\mu_{\mathbb{C}}(B, i, j) = -\zeta_{\mathbb{C}}$, σ: 1-1, τ: onto.

$$\begin{tikzcd}
 & \mathcal{F}(v, w\,;\, v - v') \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
\mathcal{L}(v', w) & & \mathcal{L}(v, w)
\end{tikzcd}$$

Hecke correspond.

$\leftarrow$ moduli of $\left\{\begin{array}{l} \text{VB } E/\widetilde{\mathbb{C}^2/\Gamma} \ + \\ \text{subshf. } \mathcal{S} \leq E|_{\text{exceptional locus.}} \end{array}\right.$

i.e. $(\underbrace{(B, 0, j)}_{\in H^s_\zeta}, C)$

$\left(\because \ (p_3 p_2)^{-1}(\Lambda(v, w) \cap H^s_\zeta) \subseteq p_1^{-1}(\Lambda(v'', 0) \times (\Lambda(v', w) \cap H^s_\zeta)) \right)$

- π_1-fiber $\simeq \mathbb{CP}^{d'-1}$ $\quad d' = w_k - \sum_\ell a_{k\ell} v'_\ell - v'_k - \dim \sum_{\substack{h \in H \\ in(h) = h}} \mathrm{im}(B'_h)$

π_2-fiber $\simeq \mathbb{CP}^{d-1}$ $\quad d = v_k - \dim \sum_{h \in H:\, in(h) = k} \mathrm{im}(B_h)$

$1 \leq k \leq n$ $\qquad\qquad$ $u = w - Cv$

$$M(\mathcal{L}(v,w)) \underset{F_k = \pi_{2!}\pi_1^*}{\overset{E_k = \pi_{1!}\pi_2^*}{\rightleftarrows}} M(\mathcal{L}(v-e^k, w)) \circlearrowright H_k = u_k \cdot$$

☆ (Lusztig, Nakajima) $\quad \forall$ fix $w \in \mathbb{Z}^n_{\geq 0}$

$$\mathcal{U}(\mathfrak{g}) \curvearrowright \bigoplus M(\mathcal{L}(v,w))$$

i.e. $[H_k, H_l] = 0 \qquad [E_k, F_l] = \delta_{kl} H_k$

$[H_k, E_l] = C_{kl} E_l \qquad [H_k, F_l] = -C_{kl} F_l$

$k \neq l$, $\displaystyle\sum_{p=0}^{1-C_{kl}} (-1)^p \binom{1-C_{kl}}{p} E_k^p E_l E_k^{1-C_{kl}-p} = 0$, same for F's.

$$\chi : \mathcal{L}(0,w) \xrightarrow{\text{constructible}} \mathbb{Q} \quad \text{w/} \quad \chi|_{\substack{\text{open}\\ \text{dense}}} \equiv 1$$

Consider $L(w) := \mathcal{U}(\mathfrak{g}^-) \cdot \chi \subset \bigoplus_v M(\mathcal{L}(v,w))$

$$= \bigoplus_v \underbrace{L(w) \cap M(\mathcal{L}(v,w))}_{L(v,w)}$$

(Nakajima) $\mathfrak{g} \curvearrowright L(w)$ irred. h.t. wt. w integ. rep.

w/ wt space $L(v,w)$ of wt. $w - Cv$.

Define $\mathcal{L}(v,w) \supseteq Y$ irred. comp.

$T_Y : L(v,w) \xrightarrow{\text{linear}} \mathbb{Q}$

$f \longmapsto$ (const) value on open dense subset of Y.

$\rightsquigarrow \Phi : L(v,w) \longrightarrow \mathbb{Q}^{\operatorname{Irr}\mathcal{L}(v,w)} \simeq H^{\mathrm{mid}}(\mathfrak{M}_g(v,w), \mathbb{Q})$

(Lusztig) (1) Φ : onto.

(2) ADE or $\hat{A}\hat{D}\hat{E} \Rightarrow \Phi$: isom.

Cor. ADE or $\hat{A}\hat{D}\hat{E}$, $\exists$ basis B of $\mathcal{U}(\mathfrak{g})$ s.t.

$\forall$ irred ht. wt. integ rep. L

$$\pi : \mathcal{U}(\mathfrak{g})^- \longrightarrow L$$
$$\alpha \longmapsto \alpha \cdot x$$

ht. wt. vector

$\pi(B \setminus (B \cap \pi^{-1}(0)))$ is a basis of L

Canonical basis.

§11. Representation of $\mathcal{U}_q(\mathfrak{g})$ [Skip].

Nakajima, Heisenberg alg. & $S^{[n]}X^2$

1995.

- Heisenberg alg. $\mathfrak{s} = \mathbb{C}\langle p_i\text{'s}, q_i\text{'s}\ c\rangle$

  $[p_i, p_j] = 0 = [q_i, q_j]\ ,\ [p_i, q_j] = \delta_{ij}\, c$

  $\mathfrak{s}$-mod: $\quad \forall\ a \in \mathbb{C}^\times$

  $\mathfrak{s} \curvearrowright \mathbb{C}[x_1, x_2, \dots] =: R \quad$ ht. wt vector 1

  $p_i = a\frac{\partial}{\partial x_i}\ ,\quad q_i = x_i\ ,\quad c = a\,.$

  $\mathfrak{s} \rtimes \mathbb{C}\langle d_0\rangle$ by $[d_0, q_j] = j\,q_j\ ,\ [d_0, p_j] = -j\,p_j$

  action: $d_0 = \sum_j j\,x_j \frac{\partial}{\partial x_j}$

  $\mathrm{Tr}_R\ q^{d_0} = \prod_{j=1}^{\infty} \frac{1}{(1-q^j)}$

- Clifford alg. $C\ell = \mathbb{C}\langle \psi_i\text{'s}, \psi_i^*\text{'s}, c\rangle \curvearrowright \Lambda^* V =: F$

  also extend by d , $\mathrm{Tr}_F q^d = \prod_{j=1}^{\infty}(1+q^j)$. $\quad V = \mathbb{C}\,dx^1 \oplus \mathbb{C}\,dx^2 \oplus \dots$

use 'correspondence':

$\mathfrak{s} \curvearrowright H_\bullet(X^{[*]}) \qquad$ twist around pt

$\hat{\mathfrak{g}} \curvearrowright H_\bullet(\{\text{instantons}/\text{ALE}\}) \qquad$ twist along curve

§ X w/ dual basis C^a's & D^a's for $H_*(X)$

$E_i^a(n) \subset X^{[n-i]} \times X^{[n]}$

$\mathcal{I}_1 \supset \mathcal{I}_2$ w/ $\mathrm{Supp}(\mathcal{I}_1/\mathcal{I}_2) = p \in D^a$

$F_i^a(n) \subset X^{[n+i]} \times X^{[n]}$

$\mathcal{I}_1 \subset \mathcal{I}_2$ w/ $\mathrm{Supp}(\mathcal{I}_2/\mathcal{I}_1) = p \in C^a$

Thm. $\mathfrak{s} \otimes \mathcal{Cl} \curvearrowright H_\bullet(X^{[*]})$

$$\sum_n [E_i^a(n)] = \begin{cases} p_i & \dim C^a \text{ even} \\ \psi_i^* & \dim C^a \text{ odd.} \end{cases}$$, same for $F_i^a(n)$'s.

- Difficult to get c_i's, except $c_1 = 1$, $c_2 = -2$.

§ App. A (i) $V \simeq \mathbb{C}^n$ $W \simeq \mathbb{C}^r$

$$M := \underset{B_1,\ B_2}{\mathrm{Hom}(V,V)^{\oplus 2}} + \underset{i}{\mathrm{Hom}(W,V)} + \underset{j}{\mathrm{Hom}(V,W)}$$

For $\mathbb{CP}^2 \ni [z_0, z_1, z_2]$

$$V \otimes \mathcal{O}(-1) \xrightarrow{\sigma} (V + V + W) \otimes \mathcal{O} \xrightarrow{\tau} V \otimes \mathcal{O}(1)$$

$$\sigma^t = (B_1 z_0 - 1_V z_1 \mid B_2 z_0 - 1_V z_2 \mid j z_0)$$

$$\tau = (-B_2 z_0 + 1_V z_2 \mid B_1 z_0 - 1_V z_1 \mid i z_0)$$

$\mathbb{C}$-ADHM eqt. $[B_1, B_2] + ij = 0 \Rightarrow \tau\sigma = 0$ cpx.

- If σ 1-1, τ onto as shf. homo.

$\Rightarrow E = \mathrm{Ker}\,\tau / \mathrm{Im}\,\sigma$ torsion-free shf $/\mathbb{CP}^1$

$rk = r$, $c_1 = 0$, $c_2 = n$ & $E|_{\{z_0=0\}=:\ell} \simeq W \otimes \mathcal{O}_\ell$

- σ : 1-1 always

τ : onto iff $\nexists$ proper $S \leq V$ w/ $\begin{cases} B_k(S) \subset S, & k=1,2 \\ i(W) \subset S \end{cases}$

- Fix Hermitian str. on V, W. Fix $\zeta_{\mathbb{R}} < 0$,

  τ : onto iff $\exists\, g \in GL(V)$ (unique up to $U(V)$)

  s.t. $\mu_{\mathbb{R}}(gB_1g^{-1}, gB_2g^{-1}, gi, jg^{-1}) = -\zeta_{\mathbb{R}}$

  where $\mu_{\mathbb{R}}(B_1, B_2, i, j) := [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + ii^\dagger - j^\dagger j$

- $+$ σ 1-1 at stalks $\Rightarrow$ E VB

Thm: $\mathcal{M}_{\mathbb{P}^2}(r, 0, n)_{\text{frame at } \ell}$

$$\simeq \left\{ (B_1 B_2\, i\, j) \in \mu_{\mathbb{C}}^{-1}(0) \;\middle|\; \begin{matrix}\sigma \text{ 1-1} \\ \tau \text{ onto}\end{matrix} \right\} / GL(V)$$

$$\simeq \underbrace{\mu_{\mathbb{C}}^{-1}(0) \cap \mu_{\mathbb{R}}^{-1}(-\zeta_{\mathbb{R}}) / U(V)}_{\text{complete HK, } \dim = 4nr.} \qquad \text{(HK quotient).}$$

<u>Conversely,</u> $E / \mathbb{CP}^2 \qquad E|_\ell \cong W \otimes \mathcal{O}_\ell$

$$V := H^1(\mathbb{CP}^2, E(-2)) \simeq H^1(\mathbb{CP}^2, E(-1))$$

$(\because\ H^{\leq 1}(E(-1)|_\ell) = 0)$

$\times z_1 \rightsquigarrow B_1$
$\times z_2 \rightsquigarrow B_2$ $: V \circlearrowleft$

$0 \to \mathcal{O}(-1) \xrightarrow{z_0} \mathcal{O} \to \mathcal{O}_\ell \to 0$

$\rightsquigarrow\ i : \underbrace{H^0(\ell, E)}_{W} \longrightarrow \underbrace{H^1(\mathbb{CP}^2, E(-1))}_{V}$

$0 \to \mathcal{O}(-3) \xrightarrow{\cdot z_0} \mathcal{O}(-2) \longrightarrow \mathcal{O}_\ell(-2) \to 0$

$\rightsquigarrow\ j : \underbrace{H^1(\mathbb{CP}^2, E(-2))}_{V} \longrightarrow \underbrace{H^1(\ell, E(-2))}_{W^* \simeq W}$

☆ Remark: ALE spaces:

$\mathfrak{g} \circlearrowright H.(\mathcal{M}\{\text{torsion free shf}\})$

$\mathfrak{g} \circlearrowright H.(\mathcal{M}\{\text{VB}\})$

Grojnowski, Instantons & Affine algebras I.

(1995) $X^{2\mathbb{C}}$ sympl. surface.

§ Thm. $H^\bullet(S^{[*]}X)$ comm./co-comm. Hopf alg.

use correspondence

$$\Lambda \overset{\text{Lagr.}}{\subset} S^{[a]}X \times S^{[a+b]}X \times S^{[b]}X$$

$$0 \to A_1 \to A_2 \to A_3 \to 0$$

product $x \cdot y = (\pi_{a+b})_* \big((\pi_a, \pi_b)^* (x \otimes y) \cdot [\Lambda] \big)$

coproduct $\Delta_{ab} z = (\pi_a, \pi_b)_* \big([\Lambda] \cdot \pi_{a+b}^* z \big)$

non-degen. inner product $(x, y) = \delta_{nm} \int_{S^{[n]}X} x \cdot y$

- $H^\bullet(S^{[*]}X)$ Fock space of $H^\bullet(X)$.

§4 Fix rank c, $\bar{\mathcal{M}} = \coprod_{c_1, k} \bar{\mathcal{M}}_{c_1, k}$ moduli of Gieseker s.s. torsion-free sh. over $(X^2, H \gg 0)$.

$C = 1 \Rightarrow \bar{\mathcal{M}} = \bigoplus_{\lambda \in H^2(X, \mathbb{Z})} (S^{[*]}X) e^\lambda$ (assume $\pi_1(X) = 0$, $w_2(X) = 0$)

$H^2(X, \mathbb{Z})$ ⊗ line bdl.

Thm: $\mathcal{F} := H^\bullet(S^{[*]}X) \otimes \mathbb{C}\{H^*(X, \mathbb{Z})\}$ vertex algebra

(eg. ALE case $\Rightarrow$ $\hat{\mathfrak{g}} \curvearrowright \mathcal{F}$ basic repr.)

§5. $c > 0$. Fix $\mathbb{C}^c \to V \to X$

$T_{V,n} := \{ \mathcal{E}/X : \text{torsion-free}, \ \mathcal{E}^{**} \simeq V \}$

(V stable $\underset{(\not\Leftarrow)}{\Longrightarrow}$ $\mathcal{E}$ stable $\Rightarrow$ $T_{V,n}$ smooth)

Thm. $H^\bullet(S^{[*]}X) \curvearrowright \bigoplus_n H^\bullet(T_{v,n})$ via corresp.

$$\Lambda \subset T_{v,a} \times T_{v,a+b} \times S^{[b]}X$$
$$0 \to A_1 \to A_2 \to A_3 \to 0$$

$\rightsquigarrow$ Heisenberg alg. $\bigoplus_{n\neq 0} H^\bullet(X,\mathbb{C})t^n \oplus \mathbb{C} \curvearrowright \bigoplus_n H^\bullet(T_{v,n})$

w/ central charge c.

§7. $\mathfrak{g}$ Kac-Moody

(Lusztig) $\mathcal{M}oduli\begin{pmatrix}\text{repr. of} \\ \text{quiver of } \mathfrak{g}\end{pmatrix} \underset{\text{Lagr.}}{\supset} \Lambda$

$\rightsquigarrow$ univ. Verma mod. for $\mathfrak{g}$

(Nakajima) $\mathcal{M}$odified quiver variety $\mathcal{M}_\zeta(w)$ ← ht wt, ζ

- generic $\zeta \Rightarrow \mathfrak{g} \overset{\text{corresp.}}{\curvearrowright} H^\wedge(\mathcal{M}_\zeta(w))$

  irred. ht. wt. w integ. $\mathfrak{g}$-mod
  ↖ cut down Verma.

- $\mathfrak{g}$ affine + $\mathrm{Tr}\,\zeta = 0 \Rightarrow \mathcal{M}_\zeta(w) = \mathcal{M}oduli\begin{pmatrix}U(n)\text{-instantons} \\ \text{over ALE } X_\zeta\end{pmatrix}$

- $U_q(\hat{\mathfrak{g}}) \curvearrowright K^{GL_w \times \mathbb{C}^\times}(\mathcal{M}_\zeta(w))$.

Kapranov–Vasserot, Kleinian Singularities, Derived categories & Hall algebras

1998

§1 finite $G \leq SL(2,\mathbb{C})$

(Gonzalez-Springberg, Verdier)

$$K_0(\widetilde{\mathbb{C}^2/G}) \simeq \mathrm{Rep}(G) \simeq \hat{h}_{\mathbb{Z}}$$

Thm. $D^b(\widetilde{\mathbb{C}^2/G}) = D^b_G(\text{mod-}\mathbb{C}[x,y])$

§ $G \curvearrowright X$

$$\begin{array}{ccc} \widetilde{X/G} & \subset & X^{[m]} \\ \downarrow & & \downarrow \\ X/G & \subset & X^{(m)} \end{array} \qquad m = |G|$$

Define

$$D^b(\mathrm{Coh}_G(X)) \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} D^b(\mathrm{Coh}(\widetilde{X/G}))$$

$$\begin{array}{ccccc} \Sigma & \overset{\text{incidence}}{\subset} & \widetilde{X/G} \times X & & \\ & \swarrow^{p_1} & & \searrow^{p_2} & \\ \widetilde{X/G} & & & & X \end{array}$$

$\Phi(-) \triangleq (Rp_{1*}(p_2^*(-) \otimes^L \mathcal{O}_\Sigma))^G$

$\Psi(-) \triangleq Rp_{2*}\, R\mathcal{H}om(\mathcal{O}_\Sigma, p_1^*(-))$

Φ, Ψ : adjoint

- Kernel for $\Phi\Psi$: $(Rp_{13*}(R\mathcal{H}om(\mathcal{O}_{\Sigma_{12}}, \mathcal{O}_{\Sigma^t_{23}})))^G$ / $\widetilde{X/G} \times \widetilde{X/G}$
- Kernel for $\Psi\Phi$: $Rs_{13*}(R\mathcal{H}om(\mathcal{O}_{\Sigma^t_{12}}, \mathcal{O}_{\Sigma_{23}}))$ / $X \times X$

  $s_{13} : X \times (\widetilde{X/G}) \times X \longrightarrow X \times X$

Theorem: $G \curvearrowright (X^2, \omega)$ sympl. surface

$\Rightarrow$ Φ, Ψ equivalences of cat.

(i.e. check: Kernel for $\Phi\Psi \sim \mathcal{O}_\Delta \subset \widetilde{X/G} \times \widetilde{X/G}$

Kernel for $\Psi\Phi \sim \mathcal{O}_{\Delta \subset X \times X} \otimes \mathbb{C}[G]$.)

(enough to check near fix pt. of subgp of G

$\rightsquigarrow G \curvearrowright \mathbb{C}^2$ [linear !])

$\rightsquigarrow$ (Koszul) free resol^{n} of $M \curvearrowleft A = \mathbb{C}[x,y]$

$$A \underset{\mathbb{C}}{\otimes} M \xrightarrow{(x,y)} (A \underset{\mathbb{C}}{\otimes} M)^{\oplus 2} \xrightarrow{\binom{y}{-x}} A \underset{\mathbb{C}}{\otimes} M$$

$\rightsquigarrow$ resd^{n} of $\mathcal{O}_\Sigma$ / $\widetilde{X/G} \times X$

Use $\begin{cases} H^0(\widetilde{X/G}, \mathcal{E}^* \otimes \mathcal{E}) = \mathbb{C}[G][x,y] \\ H^{>0}(\text{——}\,''\,\text{——}) = 0 \end{cases}$

Tauto. bdl: $\mathbb{C}^m \longrightarrow \mathcal{E} \longrightarrow X^{[m]}$ w/ $\mathcal{E}|_{[\xi]} = H^0(\xi, \mathcal{O})$

§2. $\pi \curvearrowleft G$ $\quad \mathcal{E}_\pi \searrow$

$$\widetilde{\mathbb{C}^2/G} \xrightarrow{p} \mathbb{C}^2/G$$

$$E = p^{-1}(0).$$

Dynkin	Γ^0	$\longleftrightarrow$	comp. of E	$\mathbb{P}^1_\pi$
	$\cap$			$\updownarrow$
Affine Dynkin.	Γ	$\longleftrightarrow$	irred. rep. of G	π

Lemma: $\quad \mathcal{E}_\pi|_{\mathbb{P}^1_\rho} = \begin{cases} \mathcal{O}^{\oplus d_\pi} & \pi \neq \rho \\ \mathcal{O}(1) + \mathcal{O}^{\oplus d_\pi - 1} & \pi = \rho \end{cases}$

- $H^{>0}(E, \mathcal{E}^* \otimes \mathcal{E}) = 0$

(Ito-Nakamura) $\quad G \curvearrowright \pi \not\cong 1 \Rightarrow$

(a) $\exists$ irred. $\pi', \pi'' < (x,y)/(x,y)^G$

$$\pi' \simeq \pi \simeq \pi''$$

$[\xi] \in \mathbb{P}^1_\pi \setminus \bigcup_{\rho \neq \pi} \mathbb{P}^1_\rho \Rightarrow I_\xi = I(W) \ \exists\, W \lneq \pi' \oplus \pi''$ proper G-submod.

(b) if $m_{\pi,\rho} \neq 0$ i.e. $\exists\, [\xi] \in \mathbb{P}^1_\pi \cap \mathbb{P}^1_\rho \Rightarrow I_\xi = I(\pi' \oplus \rho'')$

$G \curvearrowright \pi \rightsquigarrow \pi^!$ skyscraper w/ fiber$|_0 = \pi \,/\, \mathbb{C}^2$

$$\tilde{\pi} = \pi \otimes_{\mathbb{C}} \mathcal{O}_X \,/\, \mathbb{C}^2 \qquad (\text{VB}).$$

- π irred $\Rightarrow$ $\Phi(\tilde{\pi}) = \mathcal{E}_\pi$

  $\pi^! \sim$ Koszul $\tilde{\pi} \otimes_{\mathbb{C}} \Lambda^2 \tau \longrightarrow \tilde{\pi} \otimes_{\mathbb{C}} \tau \longrightarrow \tilde{\pi}$ $\quad G \curvearrowright \mathbb{C}^2 = \tau$

- $$\Phi(\pi^!) = \begin{cases} \mathcal{O}_{\mathbb{P}^1_\pi}(-1)[1] & \pi \neq \mathbb{C} \\ \mathcal{O}_E & \pi = \mathbb{C} \end{cases}$$

§3 Hall algebras.

$\mathcal{A}$: $\mathbb{C}$-linear Abelian category (finite type)

eg. $((\text{mod-}\mathbb{C}\text{-alg}))$, $\quad \mathrm{Coh}(\text{proj. var})$

$\mathcal{A}^{iso}(\mathbb{C}) = \mathcal{A}^{obj}/\cong$ $\qquad$ alg. stack.

Def: $H(\ ) := \mathrm{Fun}(\mathcal{A}^{iso}(\mathbb{C}))$ Hall algebra.

$f * g = p_{2*}(p_1^* f \cdot p_3^* g)$ $\qquad$ [fiber of p_2: alg. var.]

where $\{A \hookrightarrow B \mid A, B \in \mathcal{A}^{obj}\}$ $\xrightarrow{p_1} A$, $\xrightarrow{p_2} B$, $\xrightarrow{p_3} B/A$ $\quad \in \mathcal{A}^{iso}$

i.e. $A \in \mathcal{A}^{obj} \rightsquigarrow$ char. fu. $[A] \in H(\mathcal{A})$

$$[A] * [B] = \sum_{C \in \mathcal{A}^{iso}} \chi(G^C_{AB})\, [C]$$

where $G^C_{AB} \triangleq \{A' \subseteq C : A' \simeq A,\ C/A' \simeq B\}$ cpx. var.

- Γ affine type graph

$\rightsquigarrow$ Abelian category $\mathcal{R}_\Gamma$ of double repr. i.e.

vertex $i \rightsquigarrow V_i$ & edge $\{i,j\} \rightsquigarrow V_j \underset{x_{ji}}{\overset{x_{ij}}{\rightleftarrows}} V_i$ s.t. $\sum_j x_{ij} x_{ji} = 0 \ \forall i$

- finite $G \leq SL(2,\mathbb{C}) \rightsquigarrow \Gamma$

$\Rightarrow \quad \mathcal{R}_\Gamma \simeq \mathrm{Coh}_G(\mathbb{C}^2)$

(Lusztig) $\mathcal{U}(\mathfrak{g}_\Gamma^+) \longrightarrow H(\mathcal{R}_\Gamma)$ homo.

$e_i \mapsto [\underbrace{\mathbb{C}(i)}_{\text{simple obj located at vertex } i}]$

§4 Applications to surfaces.

sm. surface $S \supset C$ ADE config. of (-2)-curve $\{\mathbb{P}^1_i\}_{i\in I}$

$\mathrm{Coh}(\mathbb{C}^2/G, E) \simeq \mathrm{Coh}(S,C)$ ← coh. sh/S supp $\subseteq C$

$\big\}$ Hall

Theorem. $\mathcal{U}(\mathfrak{g}_C^+) \longrightarrow H(S,C)$ alg. homo.

$e_i \mapsto [\mathcal{O}_{\mathbb{P}^1_i}]$

Cor. $\mathcal{U}(\mathfrak{g}_C^+) \curvearrowright \mathrm{Fun}(\mathrm{Coh}(S))$.